Wirecell and Tomography

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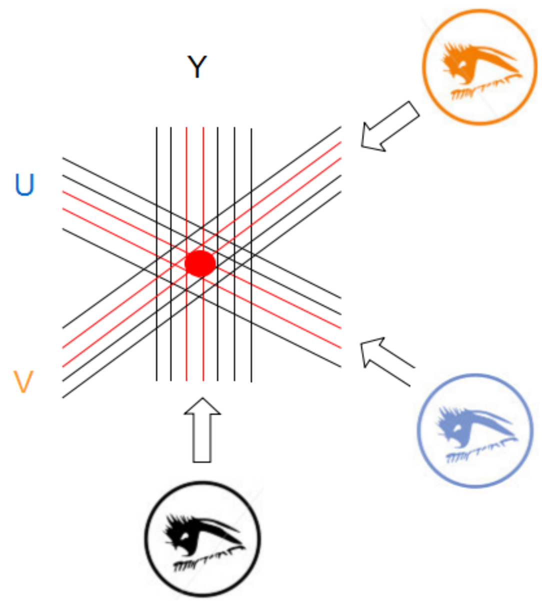


Overview

As will be shown, there is a strong connection between tomography and reconstruction of 3D objects in LArTPC, in which case the objects are three-dimensional ionization patterns. Tomography is an extremely advanced field due to a large number of important applications from medical to seismography to defense. It's a "classic" inverse problem and there is a large number of ways to approach it. A variety of mathematical techniques are used in this field, ranging from relatively straightforward to extremely advanced, especially in cases of sparse, incomplete and/or noisy data.

There are currently a number of techniques under development in DUNE with different degrees of development, maturity and performance. Wirecell presents an approach which stresses tomographic approach to reconstruction as opposed to the more typical track-finding techniques inherited from HEP. It therefore appears beneficial to have a brief survey in order to:

- Find parallels between Wirecell and some of existing techniques, as a validation exercise
- Try to identify methods that can further improve Wirecell (if possible)

In the interest of time, most of involved mathematical detail will be skipped and only general ideas presented. This survey has only started and is work in progress.

A Few Random Facts

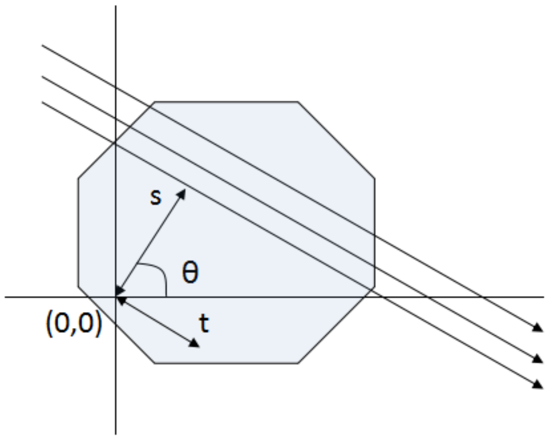
- Computed Tomography gained wide acceptance in 1970s in the form of “X-Ray CAT scan”.
- The version of CT which was initially popular was based on “Radon transform” introduced decades earlier. Ironically, on of the proponents of CT essentially derived the transform independently and encountered original papers by Johann Radon later.
- Multiplanar reconstruction (MPR) is the simplest method of reconstruction. A volume is built by stacking the axial slices (this sounds familiar).

“Classic” tomography: parallel beam irradiation and Radon Transform

Consider a 2D slice of the object with cross sectional coordinates (x, y), placed in a field of parallel beams (e.g. light). Absorbance along the individual beam direction is calculated as line integral: $p = \int \mu(x, y)dl$.

- Spoiler – this is the cornerstone of LArTPC connection to tomography.

Beams can be parametrized using two variables, angle of the slice and distance from the beam to origin: (θ, s). We also introduce coordinate t, in the direction of the beam (i.e. perpendicular to s). For each $\theta, p_\theta(s)$ will be a one-dimensional image of the object.



Changing coordinates from (x, y) to (s, t) – effectively frame rotation – absorbance p can be expressed as an integral:

$$p_\theta(s) = \int_{-\infty}^{\infty} \mu(s * cos\theta - t * sin\theta, s * sin\theta + t * cos\theta)dt \tag{1}$$

This is the Radon transform.

Inverse transform. Backprojection.

The Fourier Slice theorem states that the 1D Fourier Transform of the projection function is equal to the 2D Fourier Transform of the image evaluated on the line that the projection was taken on. Based on this, it can be shown that the original distribution can be inferred using an inversion of the projection function:

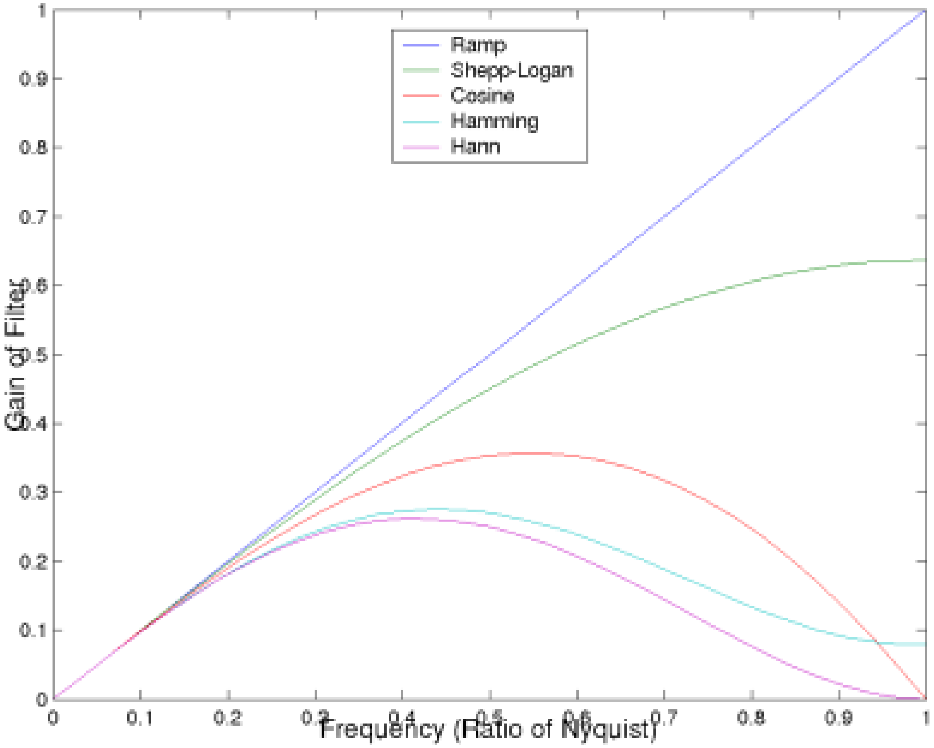
$$f_{BP}(x,y) = \int_0^\pi p(x * cos\phi + y * sin\phi, \phi)d\phi \tag{2}$$

where BP stands for "backprojection". There is a ready interpretation of the above formula: for a given point, summation of projection function values over all rays passing through that point.

It can be shown that (a) the inversion is unstable, i.e. measurement errors are amplified in the reconstructed f and (b) with finite number of measurements the problem is intrinsically ill-posed. It can also be shown that the distribution $f_{BP}(x,y)$ is a convolution of the image $f(x,y)$ and the "blurring function" $1/r$. For that reason, to arrive to the original image one needs to perform deconvolution which is done via filtering by applying a Fourier transform to p and mutiplying it by a ramp filter which is a linear function of frequency (thus emphasizing high frequencies). More on the next slide.

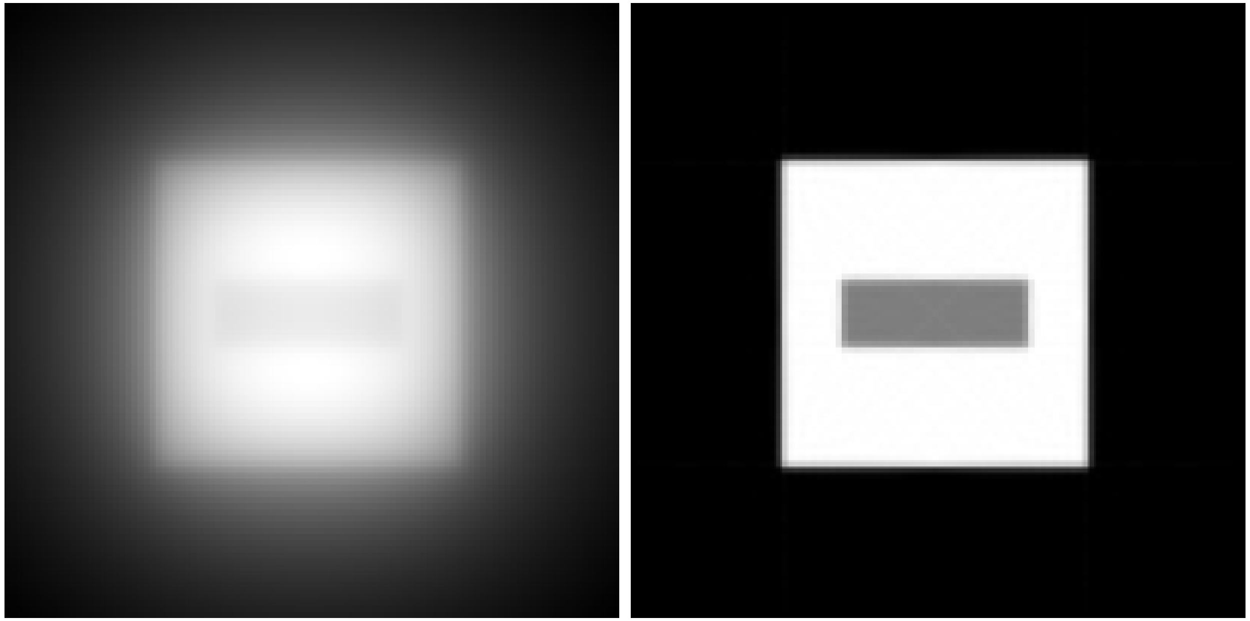
Filtering

An image reconstructed by simple inversion will typically be blurry. To correct this, a filter is applied to p via DFT with ramp filter implied by the backprojection technique. However, since it empasizes high frequencies it can also enhance noise in the image, and for that reason a variety of filters is employed in practice, depending on application. A few examples are given below.



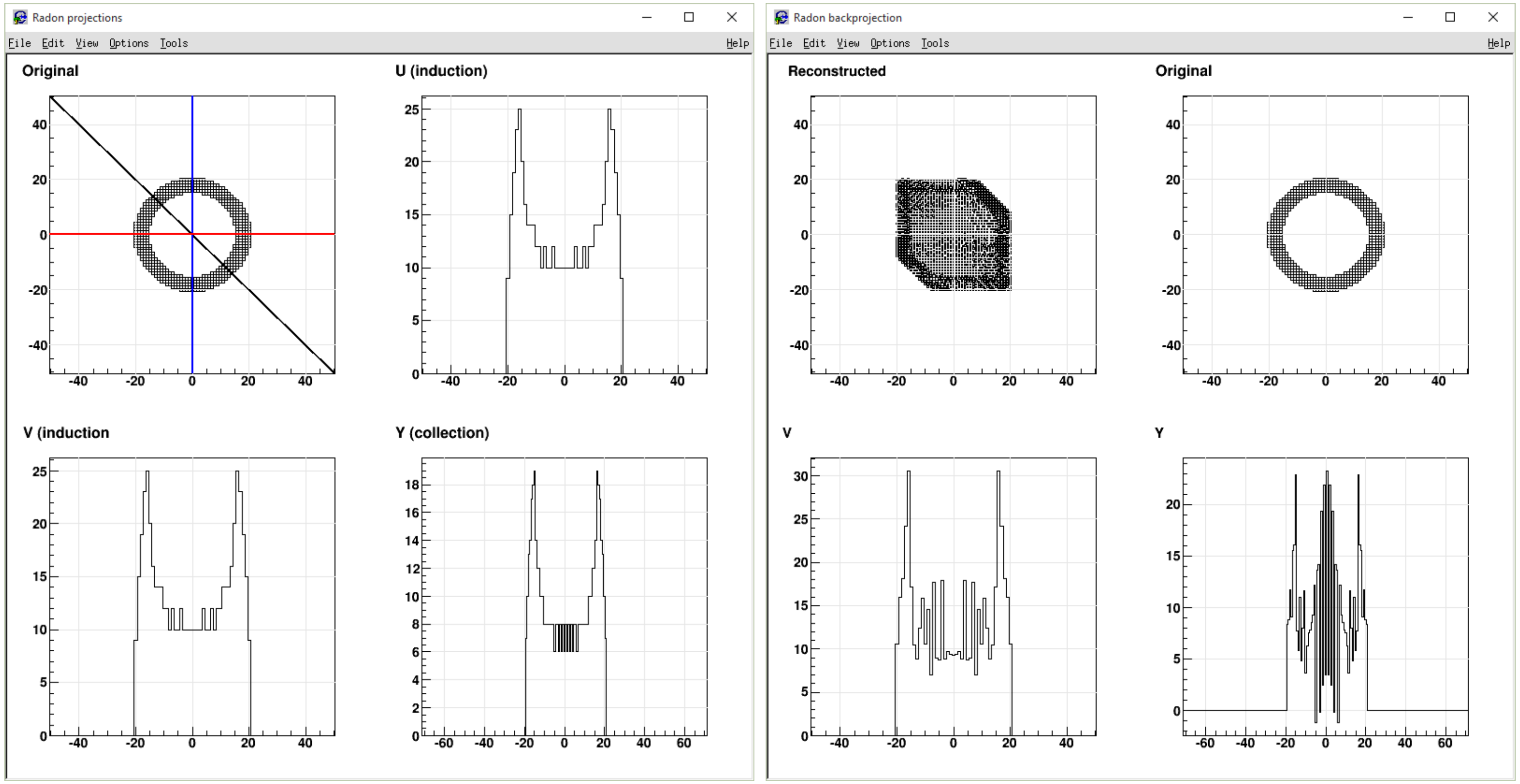
Filtering Example

An example of the effect of the filter on reconstruction is given below (direct backprojection on the left, filtered backprojection on the right).



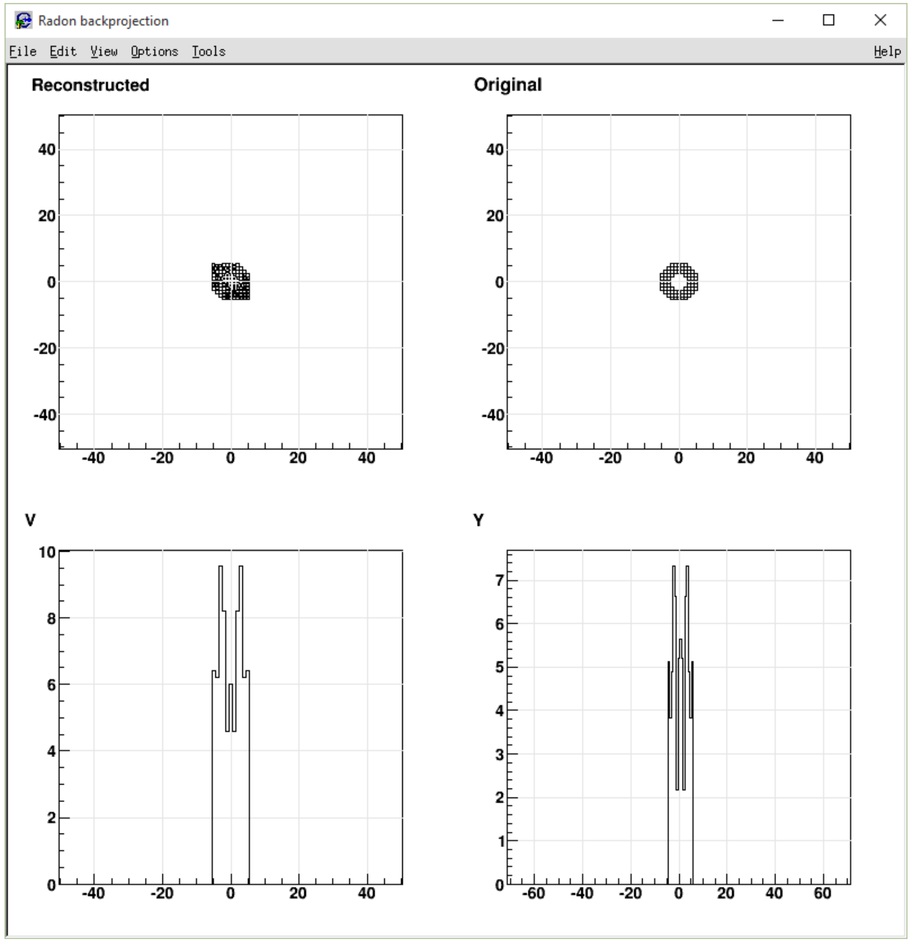
Trying it all in ROOT

To get a better feel of how these procedures work, I created a ROOT macro for discrete approximation of the Radon transform and filtered backprojection, utilizing only 3 projections in a configuration similar to (U, V, Y). To simplify code, all was done on a cartesian grid with square pixels, resulting in simulated “induction plane wires” at 45° angles to the collection wire. In the diagram below, the induciton wires are oriented along the cartesian axes and the collection wires run diagonally. Filtering was done using DFT and one of the filters presented above. For starters, a typical test image was used (a ring, which is less trivial than a solid image).



Trying to resolve finer detail

Create a ring with $R_{in} = 2$px and $R_{out} = 5$px. Amazingly, there is still a hint of the detail in the reconstructed image, also visible in its cross sections.



Comments to the exercise

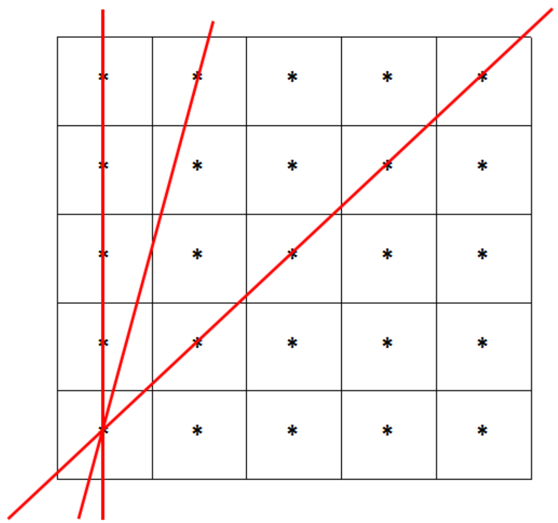
- The grid used in this exercise is equivalent to one considered in the “Wire Pattern” presentation, via a linear transformation of coordinates. The tiles then are essentially diamond shaped elements of the grid.
- Good quality recpnstruction was not expected and was not achieved. Number of projections is small in an extreme way, so the object was not correctly reconstructed, but – there was some topographic similarity.
- There is apparently some sensitivity to features on a single pixel scale.
- Will need to test it with images derived from LArTPC MC data (on the to-do list).
- It clearly works very fast.

Discrete case and the Mojette transform

The Radon transform can be considered in the discrete case (DRT). There is an application of discrete geometry which operates on pixels of the image and uses counting rules to form the summation formulas analogous to the integral in the Radon Transform. This technique is known as Mojette Transform. It operates over a set of angles satisfying the condition $\theta = tan^{-1}(q/p)$ where (q, p) are coprime numbers. One way to express the projection operator for MT is this:

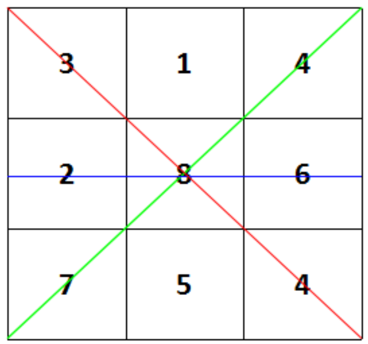
$$M_{p,q}(\rho) = \sum_x \sum_y f(x, y)\Delta(\rho - px - qy) \tag{3}$$

It implies that a pixel is included into the sum only if its center is on the projection line and presents a close analogy to the Radon transform (albeit discrete).



The restriction of angle θ leads both to a different sampling and to a different number of bins in each projection.

Mojette Inversion

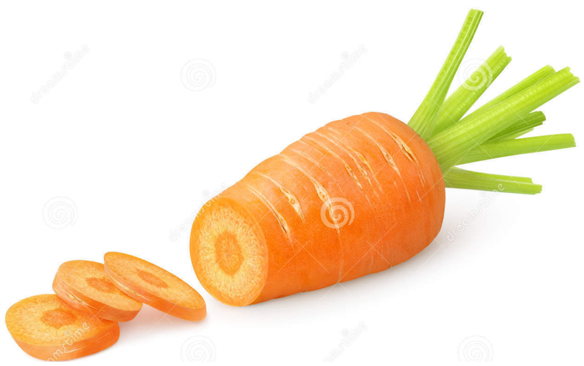


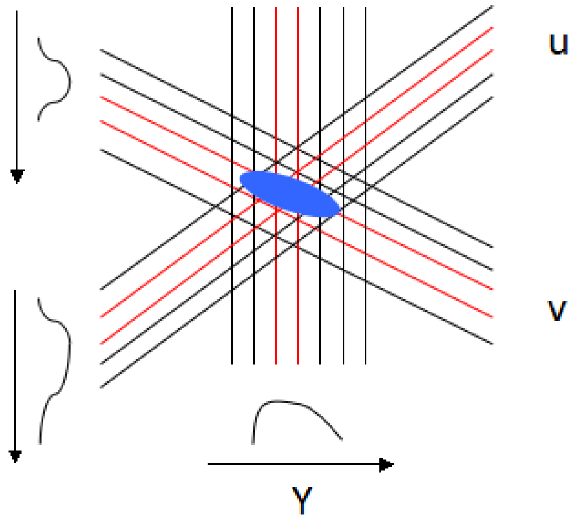
Consdier projections: R=(3,3,19,11,4), G=(7,7,15,7,4), B=(12,14,14). Starting with corners, find a well defined pixel value (meaning in a particular projection that's the only one on the ray) and subtract it from all projections. Iterate until solved.

The minimal number of projections required for precise reconstruction is given by Katz's lemma: the sum of absolute value of projection coordinate needs to be equal or larger than the dimension of the image in same direction. Note that reconstruction is done through integer arithmetic.

There are techniques for Mojette reconstruction with incomplete projection data, but these are mathematically involved and need further investigation.

Wire plane readout: similarities to tomography





The readout scheme employed in DUNE carries significant similarities to CT (specifically due to integration of charge on a single wire along the length of the wire, and qunatization of time due to ADC clock), subject to the following comments:

- A "time slice", or "time bin" in DUNE is equivalent to slices in ordinary CT, in that reconstructed 2D images in each slice need to be “stitched” together in order to obtain a 3D model of the object being interrogated
- Indeed, each wire effectively performs integration of charge along its direction, which is equivalent to a thin beam probing the absorbance of the object along the beam’s axis — with a caveat that a localized charge will produce a signal not on a single wire but on a group of wires according to laws of electrodynamics — the good new is that this can be calculated and taken into account (cf. deconvolution)
- There are (sadly!) only three projections which is very unusual for CT

Other reconstruction methods

- Filtered Backprojection (such with Radon) is one of the many reconstruction methods and it may not be optimal for every application domain. However, LArTPC image slices are pretty unusual compared to other objects typically reconstructed in tomography (i.e. there are sparse sets of relatively small track slices) so a little additional investigation is required for the final verdict.

- There is a family of reconstruction methods which maximize likelihood or expectation value of some objective function. Wirecell belongs in this category.

- There are mathematical approaches such as Kaczmarz method, which is an iterative procedure for solving large sparse systems of linear equations. From this, the Algebraic Reconstruction Technique and its variants were derived. This includes optimization approaches based on least squares.

- Iterative filtered backprojection:
$$x^{(n+1)} = x^{(n)} + \alpha FBP(y - Ax^{(n)})$$
where α is the step size, A is the model.

Dealing with sparse measurements

- The severity of ill-posedness is (kind of) inversely proporsional to the number of projections
- It is in fact impossible to do decent reconstruction in a slice without invoking some sort of apriori knowledge about the object being interrogated
- For example, a *cost function* (or a *"penalty function"*, depending on definition) is a common tool in tomography, and it does imply such knowledge
  - Sometimes likelihood is considered a part of the cost function
- One formulation: "Find image x that best fits the sinogram data y according to the physics model (detector), statistical model and prior information about the object"
  - NB. it's not the same as pattern recognition
- Certain types of tomography assume *"object sparsity"*, and that may be appropriate for LArTPC since for the most part images should not containlarge continuous features (well they might when the track is almost parallel to the plane, but otherwise...)
- There is a statistical approach whereby statistical characteristics of the family of objects being observed are used to attribute likelihood to candidate solutions

Summary

Maximum-likelihood is a technique previously utilized in absorption and emission tomography. Wirecell implements an algorithm based on similar principles in the wire-plane readout setting. It apparently is using a hypothesis of object continuity (since it groups adjacent cells) for regularization.

There are a number of complementary techniques such as iterative reconstruction that could prove useful as a possible enhancement to Wirecell and similar reconstruction techniques.